

A lexical tree for the middle-levels graphs

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ABSTRACT. Let $m = 2k + 1 > 1$ be an odd integer and let H_m be the Hasse diagram of the Boolean lattice on the coordinate set \mathbf{Z}_m . A rooted binary tree T is proposed that has as its nodes the translation classes mod m of the weight- k vertices of all H_m , for $0 < k \in \mathbf{Z}$, with an alternative form of T whose nodes are the translation classes mod m of weight- $(k + 1)$ vertices via complemented reversals of the former class representatives, both forms of T expressible uniquely via the lexical 1-factorizations of the middle-levels graphs.

1. Introduction.

Let $1 < n \in \mathbf{Z}$. The n -cube graph H_n is the Hasse diagram of the Boolean lattice on the coordinate set $[n] = \{0, \dots, n - 1\}$. Vertices of H_n are cited in three different ways interchangeably:

(a) as the subsets $A = \{a_0, a_1, \dots, a_{r-1}\} = a_0 a_1 \dots a_{r-1}$ of $[n]$ they stand for, where $0 \leq r \leq n$;

(b) as the characteristic n -vectors $B_A = (b_0, b_1, \dots, b_{n-1}) = b_0 b_1 \dots b_{n-1}$ over the field $F_2 = \{0, 1\}$ that the subsets A represent, given by $b_i = 1$ if and only if $i \in A$, ($i \in [n]$);

(c) as the polynomials $\beta_A(x) = b_0 + b_1 x + \dots + b_{n-1} x^{n-1}$ associated to the vectors B_A . A subset A as above is said to be the *support* of the vector B_A .

For each $j \in [n]$, the j -level L_j is the vertex subset in H_n formed by those $A \subseteq [n]$ with $|A| = j$. For $1 \leq k \in \mathbf{Z}$, the *middle-levels graph* M_k is defined as the subgraph of $H_m = H_{2k+1}$ induced by $L_k \cup L_{k+1}$.

Hável [2] conjectured that M_k is hamiltonian, for every $1 < k \in \mathbf{Z}$. The latest partial update on this conjecture is due to Shields et al. [6], that announced Hamilton cycles in M_{16} and M_{17} . Johnson [4] proved that M_k has a cycle of length $(1 - o(1))$ times the number of vertices, where the term $o(1)$ is of the form c/\sqrt{k} . Horak et al. [3] proved that the prism over each M_k is hamiltonian.

Two different 1-factorizations of M_k were studied in the literature: the lexical 1-factorization of Kierstead and Trotter [5], useful in our presentation below, and the modular 1-factorization of Duffus et al. [1].

In the absence of a full answer to Hável's conjecture, reduced graphs R_k of the M_k are combined with lexical 1-factorizations [5] into a rooted binary tree T with $V(T) = \cup_{k=1}^{\infty} V(R_k)$ whose structure is ruled by Catalan's triangle. The cited 1-factorizations are compatible with the reduced graphs, but the modular 1-factorizations of [1] are not, as commented between parentheses just before Theorem 2, below.

1.1. Quotient graph M_k/π of M_k . The following relation π is defined in $V(M_k)$ with elements seen as the polynomials in (c) above:

$$\beta_A(x)\pi\beta_{A'}(x) \iff \exists i \in \mathbf{Z} \text{ such that } \beta_{A'}(x) \equiv x^i \beta_A(x) \pmod{1+x^n}.$$

It is easy to see that π is an equivalence relation and that there exists a well-defined quotient graph M_k/π . For example, M_2/π is the domain of the graph map γ_2 in Figure 1, where $V(M_2/\pi) = L_2/\pi \cup L_3/\pi$, with

$$L_2/\pi = \{(00011), (00101)\}, \quad L_3/\pi = \{(00111), (01011)\}$$

and the π -classes, expressed between parentheses around one of its representatives expressed as in (b) above, composed as follows:

$$\begin{aligned} (00011) &= \{00011, 10001, 11000, 01100, 00110\}, & (00101) &= \{00101, 10010, 01001, 10100, 01010\}, \\ (00111) &= \{00111, 10011, 11001, 11100, 01110\}, & (01011) &= \{01011, 10101, 11010, 01101, 10110\}, \end{aligned}$$

showing the ten elements of L_2 contained between both pairs of braces on top, and those of L_3 likewise on the bottom row.

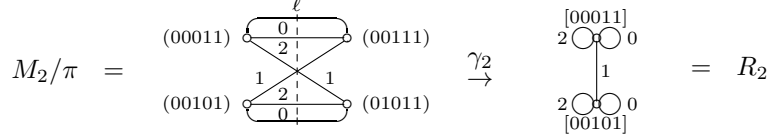


FIGURE 1. Graph map γ_2

In a way similar to this example, but now for any $k \geq 2$, we want to distribute the vertices of the vertically listed parts L_k/π and L_{k+1}/π of M_k/π (as well as those of L_k and L_{k+1} , of M_k , preserved in a separate listing) into pairs, each pair displayed on an horizontal line, with its two vertices reflected on an imaginary middle vertical line ℓ , like the dashed line ℓ in the representation of M_2/π above. To specify the sought distribution of vertices of M_k , let $\aleph : L_k \rightarrow L_{k+1}$ be the bijection given by $\aleph(b_0 b_1 \dots b_{n-1}) = \bar{b}_{n-1} \dots \bar{b}_1 \bar{b}_0$, where $\bar{1} = 0$ and $\bar{0} = 1$. Let us take each horizontal pair of vertices in our sought distribution, ordered from left to right, of the form $(B_A, \aleph(B_A))$. If $\rho_i : L_i \rightarrow L_i/\pi$ is canonical projection for both $i = k$ and $k+1$, then $\rho_{k+1}\aleph = \aleph\rho_k$. This yields a quotient bijection $\aleph_\pi : L_k/\pi \rightarrow L_{k+1}/\pi$ given by $\aleph_\pi((b_0 b_1 \dots b_{n-1})) = (\bar{b}_{n-1} \dots \bar{b}_1 \bar{b}_0)$. As a result, we have the following statement, to be proved in Section 3, where a *skew edge* is understood as a non-horizontal edge of M_k , or of M_k/π , in our adopted representation.

THEOREM 1. *Each skew edge $e = (B_{A_1})(B_{A_2})$ of M_k/π , where $|A_1| = k$ and $|A_2| = k+1$, is accompanied by another skew edge $\aleph_\pi((B_{A_1}))\aleph_\pi^{-1}((B_{A_2}))$, which is obtained from e by reflection on the vertical line ℓ equidistant from both $(B_{A_i}) \in L_k/\pi$ and $\aleph((B_{A_i})) \in L_{k+1}/\pi$, for either $i = 1, 2$. Thus: (i) the skew edges of M_k appear in pairs, the edges of each pair having their end-vertices forming two pairs of horizontal vertices equidistant from ℓ ; (ii) the horizontal edges of M_k/π have multiplicity ≤ 2 .*

1.2. Reduced graph R_k . The quotient graph R_k of M_k/π cited prior to Subsection 1.1 is obtained by denoting each horizontal pair $((B_A), \aleph_\pi((B_A)))$ in M_k/π by means of the notation $[B_A]$, where $|A| = k$. Then the vertices of R_k are the pairs $[B_A]$. In addition, R_k has:

- (1) an edge $[B_A][B_{A'}]$ per skew-edge pair $\{(B_A)\aleph_\pi((B_{A'})), (B_{A'})\aleph_\pi((B_A))\}$,
- (2) a loop at $[B_A]$ per horizontal edge $(B_A)\aleph_\pi((B_A))$.

Let $\gamma_k : M_k/\pi \rightarrow R_k$ be the corresponding quotient graph map. For example, R_2 is represented as the image of the graph map γ_2 depicted in Figure 1. Observe that R_2 contains two loops per vertex and just one (vertical) edge. The representation of M_2/π on its left has its edges indicated with colors 0,1,2, as shown near its edges in Figure 1.

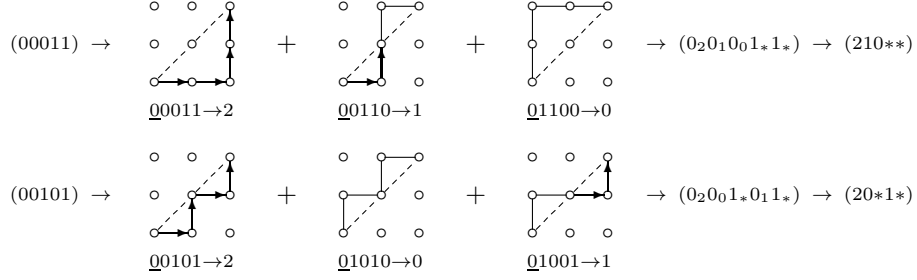
In general, each vertex v of L_k/π will have its incident edges indicated with colors $0, 1, \dots, k$ as in [5], for example by means of the following procedure, so that L_k/π admits a $(k+1)$ -edge-coloring with *color palette* $[k+1]$.

1.3. Lexical Procedure [5]. For each $v \in L_k/\pi$, there are $k+1$ n -vectors $b_0b_1 \dots b_{n-1} = 0b_1 \dots b_{n-1}$ that represent v with $b_0 = 0$. For each such an n -vector, take a grid $\Gamma = P_{k+1} \square P_{k+1}$, where P_{k+1} is the graph induced by $[k+1]$ in the unit-distance graph of \mathbf{Z} . Trace the diagonal Δ of Γ from vertex $(0,0)$ to vertex (k,k) . (Δ is represented via dashed lines, as in the instances of Figure 2, for $k=2$). Consider a stepwise increasing index $i \in \mathbf{Z}$ and an accompanying traveling vertex w in Γ initialized respectively at $i=1$ and at $w=(0,0)$. Proceed with a selection of arcs in Γ as follows:

- (1) (a) if $b_i = 0$, then select the arc $(w, w') = (w, w + (1, 0))$;
 (b) if $b_i = 1$, then select the arc $(w, w') = (w, w + (0, 1))$;
- (2) let $i := i + 1$ and $w := w'$;
- (3) repeat step (1) until $w' = (k, k)$ is fulfilled.

Consider a vertex \bar{v} of L_{k+1}/π incident to a vertex $v \in L_k/\pi$ as above. Assume that \bar{v} is obtained from a representative n -vector $b_0b_1 \dots b_{n-1} = 0b_1 \dots b_{n-1}$ of v by the sole complementation of its entry $b_0 = 0$, that is by replacing the entry b_0 of v by an entry $\bar{b}_0 = 1$ in \bar{v} , (keeping all others entries b_i of v unchanged in \bar{v} , for $i > 0$). Then, the edge $v\bar{v}$ is assigned the color given as the number of selected horizontal arcs below the diagonal Δ in Γ . According to [5], this color is unique among the $k+1$ colors $0, 1, \dots, k$ of edges incident to v . Moreover, this defines a 1-1 correspondence between $[k+1]$ and the set of edges incident to v in L_k/π .

1.4. Colorful notation for $V(M_k/\pi)$. To establish a colorful notation $\delta(v)$ for each vertex v in L_k/π , we start by representing the color assignment above, for $k=2$, as in Figure 2, where the Lexical Procedure is indicated by means of arrows (\rightarrow) from left to right, first departing from $v = (00011)$, (top), or from $v = (00101)$, (bottom), on the left side, then going to the right by depicting working sketches of $V(\Gamma)$ (separated by plus signs $+$), for each one of the three representatives $b_0b_1 \dots b_{n-1} = 0b_1 \dots b_{n-1}$ (shown as a subtitle to each sketch, with the entry $b_0 = 0$ underlined), in which to trace the selected arcs of Γ , and finally pointing, via a right arrow departing from the representative $b_0b_1 \dots b_{n-1} = 0b_1 \dots b_{n-1}$ in each sketch subtitle, the number of horizontal selected arcs lying below Δ . Only selected arcs are traced over each sketch of $V(\Gamma)$: those below Δ are indicated by means of arrows, the remaining ones, just by segments.

FIGURE 2. Representing the color assignment for $k = 2$

In each one of the two depicted examples, to the right of the three sketches and indicated by arrows, we have written a non-parenthetical modification of the notation $(b_0b_1 \dots b_{n-1})$ of v , obtained by setting as a subindex of each entry 0 the color obtained in a corresponding sketch of it, and a star $*$ for each entry 1. Still to the right of this subindexed modification of v , we have written the string of length n formed by the subindexes alone, in the order they appear from left to right. We will indicate this final notation by $\delta(v)$.

A similar pictorial argument for any $k > 2$ provides a colorful notation $\delta(v)$ for any $v \in L_k/\pi$. Each pair of skew edges $(B_A)\aleph_\pi((B_{A'}))$ and $(B_{A'})\aleph_\pi((B_A))$ in M_k/π is said to be a *skew specular edge pair*. It is not difficult to see that an argument as above provides a similar colorful notation for any $v \in L_{k+1}/\pi$ and such that:

- (1) each edge receives the same color regardless of the end-vertex of it on which the Lexical Procedure above or its modification for $v \in L_{k+1}/\pi$ is applied
- (2) and each skew specular edge pair in M_k/π receives a unique color in $[k+1]$;

only that for example for $k = 2$, in Figure 2 we have to replace each v by $\aleph_\pi(v)$, so that on the left side of Figure 2 we would have now $((00111)$, (top), and (01011) , bottom, with sketch subtitles respectively given by

$$\begin{array}{lll} 0011\bar{1} \rightarrow 2, & 1001\bar{1} \rightarrow 1, & 1100\bar{1} \rightarrow 0, \\ 0101\bar{1} \rightarrow 2, & 1010\bar{1} \rightarrow 0, & 0110\bar{1} \rightarrow 1, \end{array}$$

resulting in the same sketches in Figure 2 when the rules of the Lexical Procedure are taken with right-to-left reading and processing of the entries on the left side of the subtitles, where the roles played by the values of each b_i are now complemented; also, the subindexes after the arrows on the right of the sketches are reversed in their orientation with respect to those in Figure 2.

Since a skew specular edge pair determines a unique edge of R_k (and vice versa), the same color received by this pair can be attributed to such an edge of R_k . Of course, each vertex of M_k , M_k/π and R_k defines a bijection between its incident edges and the color palette $[k+1]$. The edges obtained via \aleph from these incident edges have the same corresponding colors, a phenomenon arising from the Lexical Procedure (and that cannot be obtained with the modular 1-factorization of $[1]$, where a modular color in its own color palette $\{1, \dots, k+1\}$ can be attributed to each arc, with opposite arcs of an edge having opposite colors, meaning that they add up to $k+2$).

THEOREM 2. *A 1-factorization of M_k/π formed by the edge colors $0, 1, \dots, k$ is obtained via the Lexical Procedure. This 1-factorization can be lifted to a covering 1-factorization of M_k and also collapsed to a quotient 1-factorization of R_k .*

PROOF. Each skew specular edge pair in M_k/π has its edges with the same color in $[k+1]$, as pointed out in item (2) above. Thus, the $[k+1]$ -coloring of M_k/π induces a well-defined $[k+1]$ -coloring of R_k . This gives the claimed collapsing to a quotient 1-factorization of R_k . The lifting to a covering 1-factorization in M_k is immediate. \square

In the forthcoming section, we use the colorful notation $\delta(v)$ established for the vertices v of R_k without enclosing the notation either between parentheses or brackets.

2. The lexical tree and its properties.

2.1. Lexical Tree T . We recall that R_1 is formed by the only vertex $\delta(001) = 10*$ and contend this to be the root of the binary tree T claimed before Subsection 1.1, that has $V(T) = \cup_{k=1}^{\infty} V(R_k)$. Such a T is defined as follows, where the concatenation of two strings X and Y is indicated $X|Y$ and $\|X\| = \text{length of } X$:

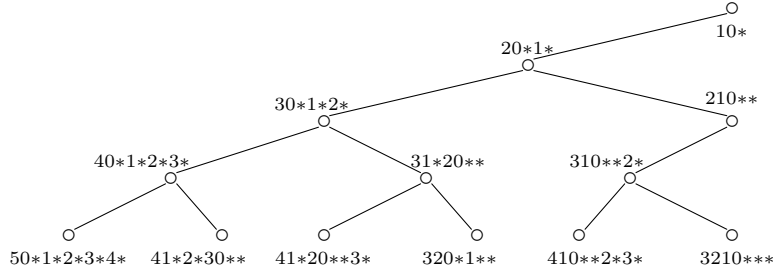


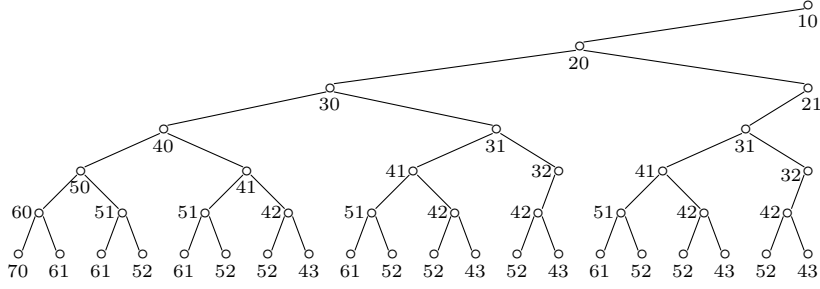
FIGURE 3. Restriction of T to its five initial levels

- (1) the root of T is $10*$;
- (2) the left child of a node $\delta(v) = k|X$ in T with $\|X\| = 2k$ is $(k+1)|X|k|*$;
- (3) the right child of a node $\delta(v) = k|X|Y|*$, where X and Y are strings respectively starting with $j < k-1$ and $j+1$, is $k|Y|X|*$;
- (4) if $\delta(v) = k|k-1|X$, then $\delta(v)$ does not have a right child.

The restriction of T to its five initial levels looks like as in Figure 3.

2.2. Alternative notations for the nodes of T . First, T has a simplified equivalent form T' obtained by indicating each node v of T by the ordered pair $\delta'(v)$ formed by the first two symbols of $\delta(v)$. A portion of T' larger than the one of T above looks like as in Figure 4.

A second form T'' of T is obtained by denoting each one of its nodes v by means of the finite sequence $\delta''(v)$ obtained by concatenating the second symbols of the notations $\delta(w)$ of the nodes w in the unique path from the root of T to v . This way, we have for example that the correspondence from the nodes of T to their new notation, for all nodes of T of length ≤ 3 is (indicated with backward arrows):

FIGURE 4. Portion of T' larger than the one of T

$00 \leftarrow 20*1*$	$000 \leftarrow 30*1*2*$	$0000 \leftarrow 40*1*2*3*$
		$0001 \leftarrow 41*2*30**$
		$0002 \leftarrow 42*30*1**$
		$0003 \leftarrow 430*1*2**$
	$001 \leftarrow 31*20**$	$0011 \leftarrow 41*20**3*$
		$0012 \leftarrow 420**31**$
	$002 \leftarrow 320*1**$	$0013 \leftarrow 431*20***$
		$0022 \leftarrow 420*1**3*$
$01 \leftarrow 210**$	$011 \leftarrow 310**2*$	$0023 \leftarrow 4320*1***$
		$0111 \leftarrow 410**2*3*$
	$012 \leftarrow 3210***$	$0112 \leftarrow 42*310***$
		$0113 \leftarrow 4310**2**$
		$0122 \leftarrow 4210***3*$
		$0123 \leftarrow 43210****$

Furthermore, this provides an alternative notation for the vertices of R_k , obtained by replacing their notation as in T by their notation as in T'' . Thus, the following fact is observed, where Φ has its arrows reverted with respect to those above.

THEOREM 3. *Let $a_{-1} = 0$. Then there is a bijection $\Phi : \mathcal{N} \rightarrow V(T)$, where $\mathcal{N} = V(T'')$ is the set of all strings $\mathbf{a} = a_0 a_1 \dots a_{k-1}$ such that $a_{i-1} \leq a_i \leq i$ in \mathbf{Z} , for $i \in [k]$, with $1 \leq k \in \mathbf{Z}$.*

PROOF. First, notice that the root of T'' is $\mathbf{a} = a_0 = 0$ and that each \mathbf{a} can be seen as a nondecreasing integer sequence. Now, for each $\mathbf{a} \in \mathcal{N}$, $\Phi(\mathbf{a})$ is the final vertex of a path in T formed as the inductive concatenation of successive paths \mathbf{a}_i , from $i = 0$ up to $i = k - 1$, with each \mathbf{a}_i starting at the final vertex of \mathbf{a}_{i-1} if $i > 0$, and at the root of T if $i = 0$; then descending to the left just one edge and stopping if $a_i = 0$; otherwise, continuing with a right path whose length is $a_i > 0$. So, each $\mathbf{a} \in \mathcal{N}$ yields a path P from the root of T to a specific node v of T . Example: the assignments $v \rightarrow P$ in R_2, R_3 are:

$00 \rightarrow (10*, 20*1*);$
 $01 \rightarrow (10*, 20*1*, 210**);$
 $000 \rightarrow (10*, 20*1*, 30*1*2*);$
 $001 \rightarrow (10*, 20*1*, 30*1*2*, 31*20**);$
 $002 \rightarrow (10*, 20*1*, 30*1*2*, 31*20**, 320*1**);$
 $011 \rightarrow (10*, 20*1*, 210**, 310**2*);$
 $012 \rightarrow (10*, 20*1*, 210**, 310**2*, 3210***).$

Thus, each $\mathbf{a} \in \mathcal{N}$ represents a path P in T departing from its root and obtained by advancing from left to right in \mathbf{a} , starting from the first entry, 0, with each entry attained in \mathbf{a} indicating a left child w of the previously attained node in P and with any integer > 0 filling that entry indicating the number of right children in P up to the next left father w' in P , if at least one such w' remains, or until v . Clearly, the obtained assignment Φ is a bijection. \square

According to Theorem 3, T'' can be presented inductively with the following alternate definition:

- (1) the root of T is $\mathbf{a} = a_0 = 0$;
- (2) the left child $l(\mathbf{a})$ of a node $\mathbf{a} = a_0 \dots a_{k-1}$ in T'' is $l(\mathbf{a}) = \mathbf{a}|a_k = a_0 \dots a_{k-1}a_k$, where $a_k = a_{k-1}$, or $l(\mathbf{a}) = a_0 \dots a_{k-1}a_{k-1}$;
- (3) the right child $r(\mathbf{a})$ of a node $\mathbf{a} = a_0 \dots a_{k-1}$ in T'' is defined if $a_{k-1} < k-1$ and in that case is given by $r(\mathbf{a}) = a_0 \dots \hat{a}_{k-1}$, where $\hat{a}_{k-1} = 1 + a_{k-1}$.

The restriction of T'' to its five initial levels looks like as in Figure 5, with double tracing for those edges joining nodes with $k = 3$, which appear themselves as bullets.

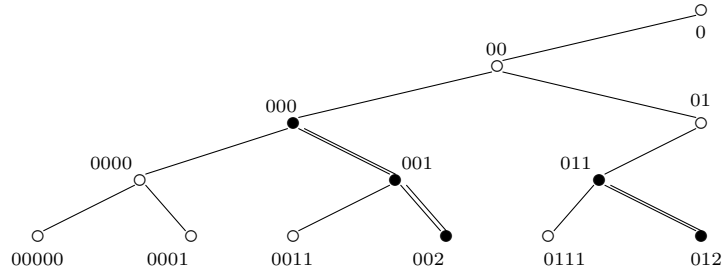


FIGURE 5. Restriction of T'' to its five initial levels

To each $\mathbf{a} = a_0 a_1 \dots a_{k-1} \in \mathcal{N}$ with $k > 1$ we associate a sequence $\psi(\mathbf{a}) = b_1 \dots b_{k-1}$ having $0 \leq b_i < i$ in \mathbf{Z} , for $0 < i < k$, and $\sum_{i=1}^{k-1} b_i < k$, by setting $b_i = a_i - a_{i-1}$, for $i = 1, \dots, k-1$. The following result can be obtained immediately.

THEOREM 4. *There exists a bijection from the set of sequences $b_1 \dots b_{k-1}$ having $0 \leq b_i < i$ in \mathbf{Z} , for $0 < i < k$, and $\sum_{i=1}^{k-1} b_i < k$, onto $V(R_k)$.*

PROOF. The bijection in the statement is the composition of ψ_k and $\Phi|V(R_k)$, where $\psi_k : \psi^{-1}(\Phi^{-1}(V(R_k))) \rightarrow \Phi^{-1}(V(R_k))$ realizes the operation ψ defined above. \square

The sequences in the statement above that map onto $V(R_2)$ are $\psi(00) = 0$ and $\psi(01) = 1$; onto $V(R_3)$ are $\psi(000) = 00$, $\psi(001) = 01$, $\psi(002) = 02$, $\psi(011) = 10$ and $\psi(012) = 11$; etc. A tree T''' obtained from T'' by denoting its root by \emptyset and any other node \mathbf{a} by $\psi(\mathbf{a})$ is partially represented in Figure 6 with stress on its subgraph induced by $V(R_3)$ as in Figure 5.

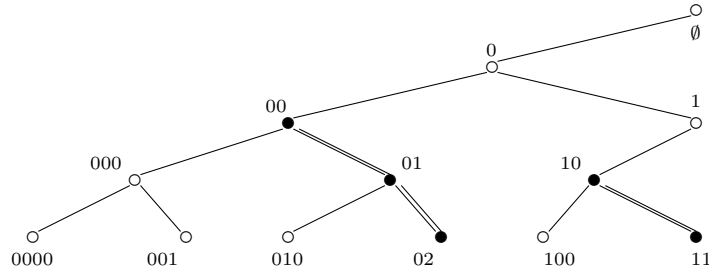


FIGURE 6. Restriction of T''' to its five initial levels

Observe that a path P from the root \emptyset to a node v of T''' represented in T'' by a string $\mathbf{a} \in \mathcal{N}$ can be traced via $\phi(\mathbf{a})$ by inspecting it from left to right: Each new inspected entry represents a left child, in the order they appear in P , from which a right path starts whose length is the number occupying that entry.

COROLLARY 5. *There exists a tree isomorphism $\Psi : T''' \rightarrow T$.*

PROOF. Ψ has underlying vertex bijection obtained as the composition

$$V(T''') \rightarrow \mathcal{N} = V(T'') \rightarrow V(T),$$

where the leftmost bijection maps root to root and restricts to the bijections ψ_k , for all $k \geq 1$, and the rightmost bijection equals Φ . Clearly, this vertex bijection extends to a tree isomorphism Ψ . \square

2.3. Linear order and adjacency table. It is clear now that in order to sort the vertices of M_k in a natural way related to its quotient graph R_k , one would parse the intersection of $T = \Psi(T''')$ and R_k via an inspection reflected in T''' by visiting its nodes in lexicographic order. This order is presented in $V(T''')$ as follows: $\emptyset, 0, 1, 00, 01, 02, 10, 11, 000, 001, 002, 003, 010, 011, 012, 020, 021, 100, 101, 102, 110, 111, \dots$. Then, for each node of T visited, the vertices v in L_k it represents are listed together with their corresponding images via \aleph . For $k = 2$, this order is given by:

01010, 10101; 00101, 01011; 10010, 10110; 01001, 01101; 10100, 11010;
00011, 00111; 10001, 01110; 11000, 11100; 01100, 11001; 00110, 10011.

An adjacency table for R_k can be obtained by having its vertices v , expressed as $\delta(v)$, heading subsequent columns, with each row containing the corresponding neighbors w , expressed as $\delta(w)$ in reverse (dictated by the employment of \aleph) and with a hat over the lexical color used in each case to obtain the corresponding adjacency of v to w . This uses an interpretation of \aleph in terms of the lexical colors of each vertex heading an adjacency column, as in the following table for $k = 3$ (on the left), where each $\delta(v)$ or $\delta(w)$ (this in reverse) is accompanied by its order of presentation in the induced graph $T[V(R_k)]$. The two rightmost columns simplify and transpose the table into a $\frac{1}{k} \binom{m}{k} \times (k+1)$ -matrix, whose columns are lexically colored in $\{0, \dots, k\}$. Notice that in this table vertices 1, 2, 5 of R_3 are incident to two different loops each. In general, R_k has exactly 2^k loops, with at least k double-looped vertices (exactly k only if k is prime).

30*1*2*	1	31*20**	2	320*1**	3	310**2*	4	3210***	5	0 1 2 3
3*2*1*0	1	3*2**01	4	3**1*02	3	3**02*1	2	3***012	5	1 3 4 1
1*023	3	**02*13	2	*2*1*03	1	*0123	5	*2**013	4	2 4 2 3
013*2	4	*1302	2	**01*23	5	*1*03*2	1	**1*023	3	3 1 5 2
*03*2*1	1	**1*023	3	**02*13	2	**013*2	4	***0123	5	4 2 5 1
										5 4 3 5

2.4. Catalan's Triangle. To count different categories of nodes of T , we recall Catalan's triangle \mathcal{T} , a triangular arrangement of positive integers, starting with:

1									
1	1								
1	2	2							
1	3	5	5						
1	4	9	14	14					
1	5	14	28	42	42				
1	6	20	48	90	132	132			
1	7	27	75	165	297	429	429		

The numbers $\tau_0^j, \tau_1^j, \dots, \tau_j^j$ in the j -th row \mathcal{T}_j of \mathcal{T} , where $0 \leq j \in \mathbf{Z}$, satisfy the following properties: **(a)** $\tau_0^j = 1$, for every $j \geq 0$; **(b)** $\tau_1^j = j$ and $\tau_j^j = \tau_{j-1}^j$,

THEOREM 6. *Level 0 of T contains just the root 10^* . The number of nodes at level $j > 0$ of T is $\binom{2k+1}{k}$ if $j = 2k + 1$, and $2\binom{2k+1}{k}$ if $j = 2k + 2$, where $k \geq 0$. For every $k \geq 1$, the number of vertices v of R_k with $\delta(v) = kjX$ is equal to τ_j^k , where $j \in [k]$. Moreover $|V(R_k)| = \tau_k^{k+1} = \tau_{k+1}^{k+1} = \text{Catalan number } \frac{1}{2k+1}\binom{2k+1}{k}$. This number is odd if and only if $k = 2^r - 1$, for $0 \leq r \in \mathbb{Z}$.*

[illegible]

\mathcal{T} determines the number of elements of R_k at each level of T by rewriting the nodes of \mathcal{T} within parentheses and preceded by the number denoting level of T :

$k=2$	1(1)	2(1)				
$k=3$	2(1)	3(2)	4(2)			
$k=4$	3(1)	4(3)	5(5)	6(5)		
$k=5$	4(1)	5(4)	6(9)	7(14)	8(14)	
...

3. Proof of Theorem 1

PROOF. With the adopted representation for the vertices of M_k , the skew edges $B_{A_1}B_{A_2}$ and $\aleph^{-1}(B_{A_2})\aleph(B_{A_1})$ of M_k are seen to be reflection of each other about the line ℓ , having their pairs of end-vertices, $(B_{A_1}, \aleph(B_{A_1}))$ and $(\aleph^{-1}(B_{A_2}), B_{A_2})$, lying each on an imaginary horizontal line of its own; that is: a line corresponding to the subset $A_1 \in L_k$ of $[n]$, for $(B_{A_1}, \aleph(B_{A_1}))$, and a line corresponding to the subset $\aleph^{-1}(B_{A_2}) \in L_k$ of $[n]$, for $(\aleph^{-1}(B_{A_2}), B_{A_2})$. On the other hand, ρ_k and

ρ_{k+1} extend together to a covering graph map $\rho : M_k \rightarrow M_k/\pi$, since the edges accompany the projections correspondingly, as for example for $k = 2$, where:

$$\begin{aligned} \aleph((00011)) &= \aleph(\{00011, 10001, 11000, 01100, 00110\}) = \{00111, 01110, 11100, 11001, 10011\} = (00111), \\ \aleph((00101)) &= \aleph(\{00101, 10010, 01001, 10100, 01010\}) = \{01011, 10110, 10110, 11010, 10101\} = (01011), \end{aligned}$$

showing the order of the elements in the images of the classes mod π through \aleph , as displayed in relation to Figure 1, presented cyclically backwards between braces, that is from right to left, continuing on the right, once one reaches a leftmost brace. Of course, this backward property holds for any $k > 2$, where

$$\aleph((b_0 \dots b_{2k})) = \aleph(\{b_0 \dots b_{2k}, b_{2k} \dots b_{2k-1}, \dots, b_1 \dots b_0\}) = \{\bar{b}_{2k} \dots \bar{b}_0, \bar{b}_{2k-1} \dots \bar{b}_{2k}, \dots, \bar{b}_1 \dots \bar{b}_0\} = (\bar{b}_{2k} \dots \bar{b}_0),$$

for any vertex $(b_0 \dots b_{2k}) \in L_k/\pi$. The projection of the skew edges of M_2 onto the only pair of skew edges of M_2/π and that of the horizontal edges of M_2 onto the two horizontal edges of M_2/π confirms the statement in this case, and it is clear that the same happens for item (i), for every $k > 2$. On the other hand, an horizontal edge of M_k/π has clearly its end-vertex in L_k/π represented by a vertex $\bar{b}_k \dots \bar{b}_1 0 b_1 \dots b_k \in L_k$, so there are 2^k such vertices in L_k , and $< 2^k$ corresponding vertices of L_k/π ; (at least $(0^{k+1}1)^k$ and $(0(01)^k)$ are end-vertices of two horizontal edges each in M_k/π). To see this implies item (ii), we will see that there cannot be more than two representatives $\bar{b}_k \dots \bar{b}_1 0 b_1 \dots b_k$ and $\bar{c}_k \dots \bar{c}_1 0 c_1 \dots c_k$ of a vertex $v \in L_k/\pi$, where $b_0 = c_0 = 0$. If for example v is represented by $d_0 \dots d_{2k} = \dots b_0 \dots c_0 \dots$, with $b_0 = d_i$, $c_0 = d_j$ and $0 < j - i \leq k$, then any *feasible* substring d_{i+1}, \dots, d_{j-1} (*feasible* to fulfill (ii) with multiplicity 2) forces in L_k/π a unique end-vertex of two horizontal edges of M_k/π , but not three. In fact, periodic continuation mod $2k + 1$ of $d_0 \dots d_{2k}$ both to the right of $d_j = c_0$ with period $\bar{d}_{j-1} \dots \bar{d}_{i+1} 1 d_{i+1} \dots d_{j-1} 0 = P_r$ and to the left of $d_i = b_0$ with period $0 d_{i+1} \dots d_{j-1} 1 \bar{d}_{j-1} \dots \bar{d}_{i+1} = P_\ell$ yields a two-way infinite string that winds up onto $(d_0 \dots d_{2k})$ to produce an end-vertex of L_k/π with two horizontal edges in M_k/π . The finite lateral periodicities of P_r and P_ℓ do not allow a third horizontal edge, up to returning back to b_0 or c_0 , (since no entry $e_0 = 0$ of $(d_0 \dots d_{2k})$ other than b_0 or c_0 is such that $(d_0 \dots d_{2k})$ has a third representative $\bar{e}_k \dots \bar{e}_1 0 e_1 \dots e_k$, besides $\bar{b}_k \dots \bar{b}_1 0 b_1 \dots b_k$ and $\bar{c}_k \dots \bar{c}_1 0 c_1 \dots c_k$). Those two horizontal edges are produced only from a feasible substring d_{i+1}, \dots, d_{j-1} . A counterexample to this and initial cases of those feasible substrings are given just below. \square

A non-feasible substring for the argument at the end of the proof above is given by $d_{i+1} d_{i+2} d_{i+3} = d_{i+1} d_{i+2} d_{j-1} = 001 = 0^2 1$. The list of feasible substrings ordered first by cardinality and then lexicographically, and accompanied by the shortest values of $n = 2k + 1$ for which they take place (between parentheses), starts with:

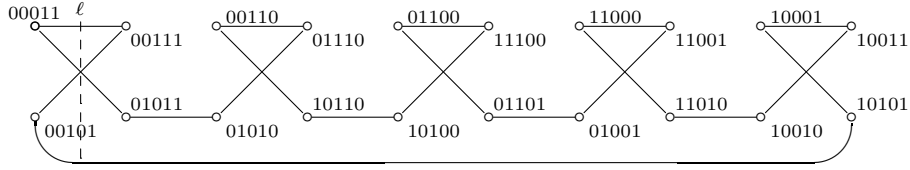
$$\begin{aligned} &(\emptyset, 5), (0, 5), (1, 3), (0^2, 7), (01, 7), (10, 7), (1^2, 7), (0^3, 9), (010, 11), (101, 13), (1^3, 15), \\ &(0^4, 11), (0^2 1^2, 15), ((01)^2, 15), (01^2 0, 17), (10^2 1, 13), ((10)^2, 15), (1^2 0^2, 15), (1^4, 19), \dots \end{aligned}$$

For example, by indicating with ‘o’ the positions $b_0 = 0$ and $c_0 = 0$ in the proof of Theorem 1, we have the following triplets of initial examples of end-vertices of two horizontal edges in L_k/π , for the first six feasible substrings in the list, with $n = 2k + 1 = 5, 7, 9; 5, 9, 13; 3, 7, 11; 7, 13, 19; 7, 13, 19$:

(1oo10)	(1o0o1)	(o1o)	(o00o111)	(o01o011)
(01oo101)	(011o0o110)	(10o1o01)	(111o00o111000)	(101o01o011010)
(101oo1010)	(10011o0o11001)	(0110o1o0110)	(000111o00o111000111)	(001101o01o011010011)

4. Succinctly encodable Hamilton cycles in M_k .

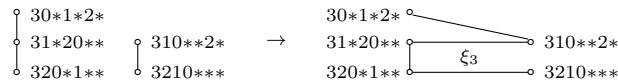
A Hamilton cycle η_k in M_k can be constructed from a Hamilton path ξ_k in R_k whose terminal vertices are $[00\dots011]$ (first zeros and then ones) and $[001\dots01]$ (alternate zeros and ones, starting with a double zero). Such a path ξ_2 in R_2 has two vertices, namely $[00011]$ and $[00101]$, with four loops altogether. First, we pull back ξ_k in R_k together with a loop at each of its terminal vertices, via γ_k^{-1} , onto a Hamilton cycle ζ_k in M_k/π . Second, ζ_k is pulled back onto a Hamilton cycle η_k in M_k by means of the freedom of selection between the two parallel horizontal edges in M_k corresponding to the two loops of one of the terminal vertices of ξ_k . In the case $k = 2$, a resulting Hamilton cycle η_2 of M_2 is represented in Figure 7, where the reflection about ℓ is used to transform ξ_2 first into a Hamilton cycle ζ_2 of M_2/π (not shown) and then into a path of length $2|V(R_2)| = 4$ starting at $00101 = x^2 + x^4$ and ending at $01010 = x + x^3$, in the same class mod $1 + x^5$, that can be repeated five times to close the desired η_2 , as shown.

FIGURE 7. Hamilton cycle in M_2

A Hamilton cycle in M_k is insured by the determination of a Hamilton path ξ_k in R_k from vertex $\delta(000\dots11) = k(k-1)\dots21 * \dots *$ to vertex $\delta(010\dots010) = k0 * 1 * 2 * \dots * (k-1) *$. (For $k = 2$, these are the only two vertices of R_2 , joined by an edge that realizes ξ_2). Observe that these two vertices in R_k are incident to two loops each, so that in general a Hamilton cycle η_k in M_k would follow by the previous remarks.

A path ξ_k as above offers the finite sequence of colors of successive edges in the path as a succinct codification for the Hamilton cycle ξ_k .

4.1. Case $k = 3$. For a fixed k , consider the induced graph $T_k = T[V(R_k)]$. Its edges descend to the right in T . In representing T_k , we trace those edges vertically, keeping the height of the levels as in T . For $k = 3$, this looks like as in Figure 8 on the left, while on the right we have traced, joining the vertices of R_3 , a Hamilton path ξ_3 with its terminal vertices incident to two loops each.

FIGURE 8. Representation of T_3

Let us analyze the Hamilton path ξ_3 depicted on R_3 in the following table:

30*1*2*	120*110*10130*	\leftrightarrow	1*03001*011*02	2213031	\leftrightarrow	1303122
310**2*	1*021*03001*01	\leftrightarrow	110*10130*120*	3223001	\leftrightarrow	1003223
31*20**	130*120*0*1011	\leftrightarrow	01001*1*021*03	3122001	\leftrightarrow	1002213
320*1**	03011*02001*1*	\leftrightarrow	0*0*10120*1113	3112023	\leftrightarrow	3202113
3210***	0*0*110*101213	\leftrightarrow	0302001*011*1*	3210023	\leftrightarrow	3200123
	030201001*1*1*	\leftrightarrow	0*0*0*10111213	3210123	\leftrightarrow	3210123
	0*0*0*10111213	\leftrightarrow	030201001*1*1*			

By translating adequately the vertices of $\xi_3 \bmod 1 + x^7$, shown vertically on the left of the table, we can see to their right a corresponding representative path ξ' in M_3 separated by double arrows (indicative of the bijection \aleph) from its image $\aleph(\xi')$. All entries 0, 1 here bear subindexes as agreed, and extensively for the images of vertices through \aleph , in its corresponding backward form. Corresponding notation for a loop is included for each of the two terminal vertices of ξ_3 before and after the data corresponding to ξ_3 and ξ' . The 6-path resulting from ξ_3 and the two terminal loops are presented in the penultimate column, by combining the non-* symbols of both vertices incident to each edge, with a hat over the coordinate in which a 0-1 switch took place, accompanied to the right by their images through \aleph .

We just extended the idea of the initial fifth (reflected about ℓ) of the Hamilton cycle η_2 in M_2 depicted previously, to the case of an initial seventh, (also reflected about ℓ), of a Hamilton cycle η_3 in M_3 . Continuing in the same fashion six more times, translating adequately $\bmod 1 + x^7$, a Hamilton cycle in M_3 is obtained. The six edges indicated on the penultimate column could be presented also with the hat positions as the leftmost ones: $\hat{0}312213$, $\hat{1}322300$, $\hat{3}122001$, $\hat{0}233112$, $\hat{1}002332$, $\hat{0}123321$. Every edge of R_k can be presented in this way. The Hamilton path ξ_3 can also be given by the sequence of hat positions: 1301, (to which 0 is prefixed and postfixed for the terminal loops). In the example for $k = 2$ above, a similar sequence for ξ_2 reduces to 1.

4.2. Case. $k = 4$. In the same way, for $k = 4$, the following sequence (of hat positions) works for a Hamilton path ξ_4 in R_4 : 1241201234032, representable as in Figure 9, where ξ_3 is also included, on top, just for comparison, with the edges of the resulting ξ_4 in R_4 drawn fully and the remaining edges of T_4 dashed, as are the edges from $V(R_3)$ to $V(R_4)$ in T . In general, for each vertex $v \in V(R_{k-1})$, there is path descending from the left child of v and continuing to the right on vertices of $V(R_k)$, for each $k > 0$, and this procedure covers all the vertices of R_k .

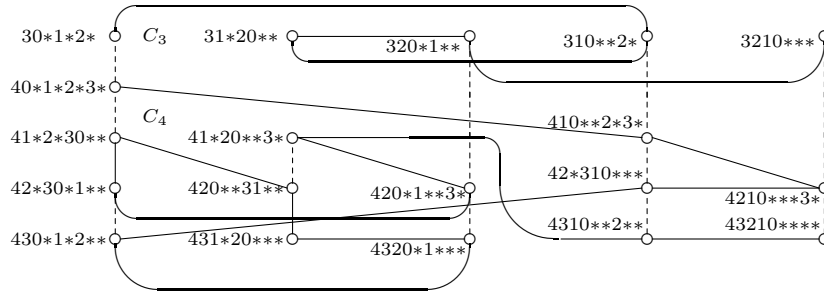


FIGURE 9. Representation of Hamilton path ξ_4

4.3. Case. $k = 5$. Let Φ_k^0 and Φ_k^1 be respectively the images, through the correspondence Φ of Theorem 3, of the smallest and largest k -sequences in the domain of Φ . (The Hamilton paths ξ_k obtained above for $k = 2, 3, 4$ started and ended respectively at Φ_k^0 and Φ_k^1). Two non-isomorphic Hamilton paths in R_5 playing the role of ξ_5 in the previous considerations about ξ_k are given by the following sequences of hat positions, where the initial and final vertices are respectively Φ_5^0 and Φ_5^1 :

15152031515052323425153545251501313531353;
40403524040503232130402010304054242024202;

so they generate corresponding non-isomorphic Hamilton cycles in M_6 , by the previous discussion.

4.4. Case. $k = 6$. Here is how to obtain 29 non-isomorphic Hamilton cycles in M_6 . They all arise from the Hamilton cycle in R_6 determined by the following cycle of hat positions, departing from Φ_1^0 and shown in a three-line display:

(5346410301615303202314304323602520101042531
53020101340341064340504012652536031501040520
412340615016560510502320616135342030636304521)

By removing the first (final) edge of this cycle, with hat position 5 (1), we obtain a Hamilton path in R_6 with final (initial) vertex Φ_6^1 incident to two loops and initial (final) vertex incident to one loop, enough to insure a Hamilton cycle in M_6 in each case. The same holds if we represent the same cycle, but starting in the second line of the display, which departs from Φ_6^0 and accounts for another pair of Hamilton cycles in M_6 . A fifth Hamilton cycle arises if we start in the third line of the display, where the first hat position corresponds to an edge with hat position 4, preceding and succeeding vertices with one and two loops, respectively.

By removing an edge with one of the following order numbers in the cycle of hat positions displayed above:

1, 28, 41, 42, 43, 44, 45, 60, 62, 100, 101, 107, 108, 96, 104, 105, 114, 122, 127, 128, 129, 130, 131, 132,

a Hamilton path in R_6 is obtained that has a loop at each one of its two terminal vertices, thus insuring a Hamilton cycle in M_6 in each case (since $2k + 1 = 13$ is prime), which yields a total of 29 non-isomorphic Hamilton cycles in M_6 . This was a list of 24 hat positions, but three of the intervening terminal vertices had two loops each, yielding a total of five new loops, which were considered above, yielding the claimed lower bound on the number of non-isomorphic Hamilton cycles of M_6 , namely 29.

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